THEORIES OF SPACE AND TIME COMPATIBLE WITH THE INERTIA PRINCIPLE

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Abstract. A general formal definition of a theory of space and time compatible with the inertia principle is given. The formal definition of reference frame and inertial equivalence between reference frames are used to construct the class of inertial frames. Then, suitable cocycle relations among the coefficients of space-time transformations between inertial frames are established. The kinematical meaning of coefficients and their reciprocity properties are discussed in some detail. Finally, a rest frame map family is introduced as the most general constitutive assumption to obtain the coefficients and to define a theory of space and time. Four meaningful examples are then presented.

1. Introduction.

The inertia principle (i.p.) is the basic postulate of all those descriptions of the physical world in which the large scale gravitational forces are neglected and the uniform motion of free particles have an "absolute" meaning.

In its traditional formulation, the i.p. asserts that an *"absolute" frame of reference* exists with respect to which free particles have a constant speed [1], [13], [17]. However, the i.p. alone is not sufficient to determine the absolute frame of reference. From the theoretical point of view, that last may be determined owing to some additional assumptions which, unfortunately, fail when compared with the experiments. For instance, Newton postulated the existence of absolute space and time on the ground of its *rationalistic theology.* Descartes, and later Leibniz, supposed that space and time are not *real things* and that space and matter coincide [1], [6-8]. On the other hand, in many theories of electricity and magnetism of XIX century, the absolute space were identified with the system respect to which the *aether* is at rest [17]. In that system, light propagates isotropically. Finally, in the second half of XX century, many authors identified as absolute the frame of reference respect to which the cosmic background radiation is isotropic [4], [9], [22], [28].

Anyway, the i.p., alone, determines a whole class of reference frames, called the *inertial frames of reference* (i.f.r.), which move, one respect to each other, with a constant dragging velocity. Therefore, the problem of determining the mathematical structure of *space-time transformations* between the frames of that class arises. That problem has been considered by the Neopositivism as well as by other different phylosophic tendencies [2], [3], [10-12], [15], [16]. It is universally accepted that a rigorous foundation of a theory of space and time requires:

a. some operative criteria for the definition of space and time measurements, togheter with an operative criterion for the synchronization of distant clocks, in any i.f.r.;

b. a definition of the deformation rates of the measures of space and time, together with a definition of the simultaneity defect, associated to any space-time transformation.

Following that line of foundational research, the present work proposes a definition of a *theory of space and time compatible with the inertia principle*, which is sufficiently general to encompass all the examples known in the literature. These examples include both those theories based on the principle of relativity (the Galilei and the Einstein-Poincaré special relativity), and those based on the existence of an absolute frame, as for example the "Lorentz relativity" [23] or the absolute theory, kinematically equivalent to special relativity [22], [25-27], [29], [30].

The paper is organized as follows. In Section 2 the formal definition of a *frame of reference* is introduced. As it is universally accepted in Epistemology [5], [10], [11], [19], in order to coordinate the events by means of space and time, three independent operative criteria must be available in any i.f.r.. The first two regulate the space and time measurements, whereas the third regulates the synchronization of clocks. As far as that last criterion is concerned, a distinction between synchronization by "clock transport" and by "first signals" has been made [5], [10], [13], [22], [25-27], [29], [30]. In both cases, a definition which is operatively correct can be obtained assigning, by a convention, the value of the "one-way" velocities of the clocks carried as well as of the first signals. Because of the motivations above, our formal definition of a frame of reference will assign, together with a time Θ and a space Σ (which are euclidean spaces of dimension one and three), the set C of the one-way velocities of material particles, which is an open neighbourhood of the zero vector in the space of the translations of Σ: $C \subseteq \mathbf{\Sigma}$. The set C is also important since it represents the domain of definition of the constitutive relations which determine the coefficients of space-time transformations as functions of the dragging velocity. Our definition is completed by the assigning of the coordination map of the events: $\phi : M \to \Theta \times \Sigma$.

Section 3 is devoted to introduce the concept of *inertial equivalence* between frames of reference. Two different frames of reference, a and b, are said to be in the inertial

equivalence if the space-time transformation between them, $\phi_{ba} = \phi_b \circ \phi_a^{-1}$, preserves the uniform motions. Inertial equivalence between a and b implies that the the space-time transformation ϕ_{ba} is an isomorphism between the affine structures of the relative times and spaces. Hence, this transformation is determined once the dragging velocity (for instance, of b with respect to a) together with three additional coefficients are given: a positive real number Δ_{ba}; a linear form defined on the space relative to a, $\tau_{ba} \in \mathbf{\Sigma}_a^*$; a linear isomorphism between the relative spaces, $\sigma_{ba} : \mathbf{\Sigma}_a \to \mathbf{\Sigma}_b$. We will show that the inertial equivalence is a relation of equivalence in the sense of set theory. Therefore, from the formal point of view, to assume i.p. is equivalent to fix a class of inertially equivalent frames of reference, I. Finally, we prove that the coefficients of the space-time transformations between frames of a same class, satisfy suitable *cocycle relations*.

In Section 4 we discuss the kinematical meaning of the coefficients of any space-time transformation, ϕ_{ba}, between inertially equivalent frames. The coefficient Δ_{ba} determines the *deformation ratio of time measures*, performed with a stationary clock in a, and with the clocks which are on its trajectory in b. The coefficient τ_{ba} determines the *simultaneity defect*: two events which are simultaneous in a, and spatially separated by a vector $\mathbf{r} \in \mathbf{\Sigma}_a$, are no longer simultaneous in b, but separated in time by the interval:

$$t_2' - t_1' =< \tau_{ba} \mid \mathbf{r} > .$$

The third coefficient, $\sigma_{ba} : \mathbf{\Sigma}_a \to \mathbf{\Sigma}_b$, represents the spatial isomorphism which associates to any stationary rod in b, $\mathbf{r}' \in \mathbf{\Sigma}_b$, its instantaneous configuration in a, $\mathbf{r} \in \mathbf{\Sigma}_a$, through the relation:

$$\mathbf{r}' = \sigma_{ba}\mathbf{r}.$$

Therefore σ_{ba} determines the *deformation ratio of the space measures*:

$$\lambda_{ba}(\mathbf{r}) =: \frac{\mid \sigma_{ba}\mathbf{r} \mid}{\mid \mathbf{r} \mid}.$$

The eigenvalues associated to σ_{ba} are said the *principal deformation ratios* of space measures, while the corresponding eigenvectors are said the *principal vectors*. By a *principal basis* we mean any oriented, orthonormal basis consisting of principal vectors. If we refer the space Σ_a to a principal basis, then we get a matrix representation of the space-time transformation ϕ_{ba} which is diagonal with respect to the space coordinates.

Inertial equivalence "per se" does not imply any relation between the principal bases, the dragging velocity and the simultaneity defect associated to a space-time transformation. However three properties of the coefficients, which are not a logical consequence of inertial equivalence, are assumed to hold in the relativistic theories as well as in the theories based on the absolute frame. The first class of theories use them to describe all the space-time

transformations, while the second to describe only the transformations between the absolute system and any other inertial frame. These properties, which we call the *reciprocity conditions*, and analyze in Section 5, can be expressed as follows:

1) the dragging velocity is a principal vector;
2) the deformation ratio of space measures is constant on the plane normal to the dragging velocity;
3) the simultaneity defect vanishes on the plane normal to the dragging velocity.

A consequence of the reciprocity conditions is that the principal bases and the simultaneity defect depend only on the dragging velocity, i.e. only on the "relative motion" of the two frames involved in the transformation. Therefore, the transformation is determined by the dragging velocity, $\mathbf{u}$, up to four real parameters: the deformation ratio of time measures, Δ_{ba}; the two principal deformation ratios of space measures, associated to the dragging velocity and its orthogonal plane:

$$\lambda =: \lambda_{ba}(\mathbf{u}) \quad ; \quad \mu =: \lambda_{ba}(\mathbf{r}) \quad , \quad \mathbf{r} \in (\mathbf{u})^{\perp};$$

and, finally, the component of the simultaneity defect in the direction of $\mathbf{u}$:

$$\theta =: < \tau_{ba} \mid u^{-2}\mathbf{u} > .$$

So, only when the reciprocity conditions hold for a space-time transformation ϕ_{ba}, then we find the familiar coordinate expression:

$$\begin{aligned} t' &= \Delta_{ba} t + u\theta x^1 \\ {x'}^1 &= \lambda(x^1 - ut) \\ {x'}^\alpha &= \mu x^\alpha \quad , \quad \alpha = 2, 3. \end{aligned}$$

Another consequence of the reciprocity conditions is their symmetry, that is, they hold for the transformation ϕ_{ba} if and only if they hold for the inverse transformation, $\phi_{ab} = \phi_{ba}^{-1}$ also. Although in literature on special relativity and its "test theories", conditions 1-3 are interpreted as a consequence of homogeneity and isotropy of the physical space [13], [15], [18], [19], [22], [25-28], it is in our opinion that they express, instead, the property that the space-time transformation could be determined as a function of the relative motion only.

Once we defined the frame of reference, introduced the inertial equivalence and fixed a class I, in order to construct a theory of space and time compatible with the inertia principle it remains to determine the explicit functional form of the coefficients of the transformations involving the frames inside the class I. To this end, the cocycle relations alone are not sufficient since they express only the *compatibility conditions* for any explicit assignment of the coefficients. In order to determine these last explicitly, we use the so called *rest frame*

principle. It states that for any fixed i.f.r. $a \in I$, any other frame in the class I is uniquely determined by its dragging velocity with respect to a. In other words, all the frames of reference in I may be described, in a, as the rest frames of free material particles. The rest frame principle, even if not mentioned in this form, is used to construct any type of space-time transformations we know. In fact, its obvious consequence is that for a given i.f.r. $a \in I$, the coefficients of any space-time transformation ϕ_{ba} are functions of the dragging velocity relative to a. In Section 6, the coefficients of space-time transformations are obtained by assigning a *rest frame map family*:

$$\mathcal{R} =: \{\mathcal{R}_a : C_a \to I / a \in I\}.$$

Any $\mathcal{R}_a \in \mathcal{R}$ is defined on the set C_a of the one-way velocities of free material particles relative to a; moreover, it is a bijection such that, according to the rest frame principle, for any $b \in I$ the vector:

$$\mathbf{u} =: \mathcal{R}_a^{-1}(b)$$

is the dragging velocity of b with respect to a. Further, the elements of $\mathcal{R}$ verify suitable compatibility conditions coming from the cocycle relations. This goal is obtained by requiring that for any pair of i.f.r. $a, b \in I$, the corresponding rest frame mappings, $\mathcal{R}_a : C_a \to I$ and $\mathcal{R}_b : C_b \to I$, be related by the law of composition of velocities:

$$\mathcal{R}_b^{-1} \circ \mathcal{R}_a(\mathbf{v}) = \frac{\sigma_{ba}(\mathbf{v} - \mathbf{u})}{\Delta_{ba} - < \tau_{ba} \mid \mathbf{v} >}.$$

The conditions of compatibility, expressed in that form, may be used both to select the constitutive functions determining the coefficients of *any* space-time transformation (like in the relativistic theories), and to obtain such constitutive functions *once those relative to a given "absolute" frame are known* (like in the theories based on an absolute frame). Section 6 is closed by defining a *theory of space and time compatible with the inertia principle* as a triplet:

$$\mathcal{T} \equiv (M; I; \mathcal{R})$$

given by the universe of events, M; the class of i.f.r., I; and a rest frame map family $\mathcal{R}$ based on I. We quote four meaningful examples in order to show that our definition encompass all the known theories. These examples are: Galileian relativity, Einstein-Poincaré's special relativity, Lorentz's relativity and the absolute theory kinematically equivalent to special relativity.

Let us conclude with a few words on the mathematical form in which we present our approach. Since the original work by Minkowski in 1908, special relativity has been formulated by introducing a Lorentz metric directly on the universe of events M, [13]. According to such a formulation, both the class I of i.f.r. and the rest frame map family $\mathcal{R}$, may be

obtained by this Lorentz metric. Recently, this approach has been extended to Galileian relativity [14], [20], [21], [24]. This formulation is based on two metrics defined on M, one for space and the other for time. At our knowledge it is not yet clear if and how a similar intrinsic geometric point of view may be extended to the theories with an absolute frame. In any case, as first noted by Poincaré [23], to these theories cannot be associated a group of transformations. Because of the motivations above, we formulated our approach by using a formalism which neither involves directly the universe of events, M, nor tries to introduce an intrinsic geometric structure on it.

2. Global space-time reference frames.

Let M be the set of events. We call M the *universe*. A *global space-time reference frame* of M consists of a splitting of M into time and space, together with the specification of the set of one-way velocities of material particles, relative to an observer. More formally, a reference frame of M is a triplet,

$$(\Theta \times \Sigma; C; \phi),$$

consisting of:

(a) a product, $\Theta \times \Sigma$, of an one-dimensional by a three-dimensional affine spaces, endowed with orientations and euclidean metrics;

(b) an open convex neighbourhood of the zero vector, with regular boundary, in the vector space of translations of Σ, $C \subseteq \mathbf{\Sigma}$;

(c) a one-to-one map, $\phi : M \to \Theta \times \Sigma$.

Θ represents the *relative time*. Euclidean structure and orientation of Θ are fixed by a non zero vector in the vector space of translations, $\mathbf{e}_0 \in \mathbf{\Theta}$, which represents the oriented unit time interval. Σ represents the *relative space*. Euclidean metric of Σ is fixed by an euclidean scalar product in the vector space of translations, $h : \mathbf{\Sigma} \times \mathbf{\Sigma} \to \mathbb{R}$. Any normal vector, $\mathbf{e} \in \mathbf{\Sigma}$, $h(\mathbf{e}, \mathbf{e}) = 1$, represents an oriented rod with unitary lenght. The orientation of Σ is fixed by a class of $SO(3)$-related orthonormal bases of $\mathbf{\Sigma}$.

C represents the set of *one-way relative velocities* of material particles. It is assumed to be open in account of a general "stability" of physical laws. This is tantamount to admit that physical laws are such that if $\mathbf{v} \in C$ is any physical velocity, then neighbourhoods of $\mathbf{v}$ can be found, all consisting of physical velocities. Convexity of C and smoothness of its boundary account for a reasonable compatibility between the description of motion and the causality relation in M.

Finally, ϕ represents the *global coordination map*, which associates any event, $x \in M$, with an instant of time and a point of space: $\phi(x) = (T, P)$. From the operational point

of view, the definition of ϕ is subordinated to some fixed rules for the measurements of time and space, together with a fixed rule for the synchronization of clocks. Different rules give rise, in general, to different theories of space and time. Space and time separations between events are defined in the obvious way, by means of ϕ; that is, if $\phi(x_1) = (T_1, P_1)$, and $\phi(x_2) = (T_2, P_2)$, then:

- the *space separation* between x_1 and x_2 is represented by the vector $\mathbf{r} =: P_2 - P_1 \in \boldsymbol{\Sigma}$, and its measure is given by the real number: $\mid \mathbf{r} \mid =: h(\mathbf{r}, \mathbf{r})^{1/2} \geq 0$;
- the *time separation* between x_1 and x_2 is represented by the vector $T_2 - T_1 \in \boldsymbol{\Theta}$, and its measure is given by the real number: $< \mathbf{e}_0^* \mid T_2 - T_1 >$, where $\mathbf{e}_0^* \in \boldsymbol{\Theta}^*$ is the dual vector of $\mathbf{e}_0$.

The events x_1 and x_2 are said to be *genidentic* [5], [10], if $P_1 = P_2$; on the other hand, they are said to be *simultaneous*, if $T_1 = T_2$.

Once an origin of time, $T_0 \in \Theta$, is fixed, then a *time scale* for the events is defined within the reference frame. Indeed, for any $x \in M$, if $\phi(x) = (T, P)$, then:

$$T = T_0 + t\mathbf{e}_0,$$

where $t =:< \mathbf{e}_0^* \mid T - T_0 >$. Now, let $T_{0'} = T_0 + t_{0'}\mathbf{e}_0$ be a different time origin, then the two time scales defined by $T_{0'}$ and T_0 are related by: $t' = t - t_{0'}$. Moreover, once an origin in space, $O \in \Sigma$, is fixed, then a *vector representation* of events is defined within the reference frame. Indeed, if $\phi(x) = (T, P)$, then:

$$P = O + \mathbf{r},$$

where $\mathbf{r} \in \boldsymbol{\Sigma}$ is the vector which translates O on P. Now, let $O' = O + \mathbf{r}_{O'}$ be a different space origin, then the two vector representations defined by O' and O are related by: $\mathbf{r}' = \mathbf{r} - \mathbf{r}_{O'}$. The singling out of both T_0 and O, gives rise to a representation of events by means of their time scale coordinates and space vectors, instead of the more formal corresponding objects in Θ and Σ. In fact, for any $x \in M$, we have:

$$\phi(x) = (T, P) = (T_0, O) + (t\mathbf{e}_0, \mathbf{r}).$$

Let now $\mathbf{v} \in C$ be any velocity; then the *uniform motions* with velocity $\mathbf{v}$ are represented in $\Theta \times \Sigma$ by the straight lines:

$$m(t) =: (T_0, O) + (t\mathbf{e}_0, P_0 - O + t\mathbf{v}) = (T_0 + t\mathbf{e}_0, P_0 + t\mathbf{v}),$$

in which T_0 and O are the origins in time and in space, and P_0 is the starting point of the motion. More generally, a motion is represented by a curve:

$$m(t) = (T_0, O) + (t\mathbf{e}_0, \mathbf{r}(t)),$$

in which $t \mapsto \mathbf{r}(t)$ is a $\mathbf{\Sigma}$-valued function such that $\dot{\mathbf{r}}(t) \in C$. Velocity and acceleration at the time t are defined by: $\mathbf{v}(t) =: \dot{\mathbf{r}}(t)$ and $\mathbf{a}(t) =: \ddot{\mathbf{r}}(t)$. Clearly, $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are invariant with respect to substitutions of the origin in time and space.

3. Inertial equivalence and space-time transformations.

Let $a =: (\Theta_a \times \Sigma_a; C_a; \phi_a)$, and let $b =: (\Theta_b \times \Sigma_b; C_b; \phi_b)$, be two reference frames. The *space-time transformation from a to b* is defined by:

$$\phi_{ba} =: \phi_b \circ \phi_a^{-1} : \Theta_a \times \Sigma_a \to \Theta_b \times \Sigma_b.$$

We say that a and b are *inertially equivalent*, and write $a \sim_I b$, if the following properties are fulfilled by ϕ_{ba}:

(a) ϕ_{ba} is a smooth diffeomorphism;

(b) there exist a smooth function, $\delta_{ba} : C_a \to \mathbb{R}^+$, and an orientation preserving smooth diffeomorphism, $\Phi_{ba} : C_a \to C_b$, such that the following equation holds:

$$\phi_{ba}(T_0 + t\mathbf{e}_0, P_0 + t\mathbf{v}) = (T_0' + \delta_{ba}(\mathbf{v})t\mathbf{e}_0', P_0' + \delta_{ba}(\mathbf{v})t\Phi_{ba}(\mathbf{v})), \tag{3.1}$$

where

$$(T_0', P_0') =: \phi_{ba}(T_0, P_0),$$

for any $(T_0, P_0, \mathbf{v}) \in \Theta_a \times \Sigma_a \times C_a$, and for any $t \in \mathbb{R}$.

In other words, inertial equivalence means that ϕ_{ba} transforms uniform motions relative to a into uniform motions relative to b. Moreover, inertial equivalence requires that if a material particle moves with constant velocity $\mathbf{v} \in C_a$ with respect to a, then its velocity with respect to b depends only on $\mathbf{v}$, by means of: $\mathbf{v}' = \Phi_{ba}(\mathbf{v})$. Finally, if $x_1, x_2 \in M$ is any pair of events connected by the propagation of this particle, that is, if:

$$\begin{array}{l} \phi_a(x_i) = (T_0 + t_i\mathbf{e}_0, P_0 + t_i\mathbf{v}) \\ \phi_b(x_i) = (T_0' + t_i'\mathbf{e}_0', P_0' + t_i'\mathbf{v}') \end{array} \quad , \quad i = 1, 2,$$

then inertial equivalence requires that the ratio of the time separation between x_1 and x_2 measured in b by that measured in a, depends only on $\mathbf{v}$ by means of:

$$\frac{t_2' - t_1'}{t_2 - t_1} = \delta_{ba}(\mathbf{v}).$$

Although the most natural way to define the inertial equivalence between the reference frame a and b is to require that the space-time transformation ϕ_{ba} preserves the uniform motions, it is a very easy consequence that ϕ_{ba} also preserves the affine structures of time and space.

Theorem 1: *If $a \sim_I b$, then the transformation ϕ_{ba} is an affine isomorphism.*

Proof. Since ϕ_{ba} is a smooth diffeomorphism, then its differential is a smooth function which takes its values in the set of linear isomorphisms between $\mathbf{\Theta}_a \oplus \mathbf{\Sigma}_a$ and $\mathbf{\Theta}_b \oplus \mathbf{\Sigma}_b$:

$$d\phi_{ba} : \Theta_a \times \Sigma_a \to (\mathbf{\Theta}_a \oplus \mathbf{\Sigma}_a)^* \otimes (\mathbf{\Theta}_b \oplus \mathbf{\Sigma}_b).$$

Now, if we represent ϕ_{ba} by the matrix:

$$d\phi_{ba} = \begin{pmatrix} \frac{\partial T'}{\partial T}\mathbf{e}_0^* \otimes \mathbf{e}_0' & \frac{\partial T'}{\partial P} \otimes \mathbf{e}_0' \\ \mathbf{e}_0^* \otimes \frac{\partial P'}{\partial T} & \frac{\partial P'}{\partial P} \end{pmatrix}$$

then we get the functions:

$$\frac{\partial T'}{\partial T} : \Theta_a \times \Sigma_a \to \mathbb{R}$$
$$\frac{\partial T'}{\partial P} : \Theta_a \times \Sigma_a \to \mathbf{\Sigma}_a^*$$
$$\frac{\partial P'}{\partial T} : \Theta_a \times \Sigma_a \to \mathbf{\Sigma}_b$$
$$\frac{\partial P'}{\partial P} : \Theta_a \times \Sigma_a \to \mathbf{\Sigma}_a^* \otimes \mathbf{\Sigma}_b,$$

which are smooth and, in addition, $\partial P'/\partial P$ takes its values in the set of orientation preserving linear isomorphisms between $\mathbf{\Sigma}_a$ and $\mathbf{\Sigma}_b$. Now, by differentiating (3.1), we find:

$$\begin{aligned} \left(\frac{\partial T'}{\partial T}\right)_{T_0,P_0} + < \left(\frac{\partial T'}{\partial P}\right)_{T_0,P_0} \mid \mathbf{v} >&= \delta_{ba}(\mathbf{v}) \\ \left(\frac{\partial P'}{\partial T}\right)_{T_0,P_0} + \left(\frac{\partial P'}{\partial P}\right)_{T_0,P_0} \mathbf{v} &= \delta_{ba}(\mathbf{v})\Phi_{ba}(\mathbf{v}), \end{aligned} \tag{3.2}$$

for any $(T_0, P_0, \mathbf{v}) \in \Theta_a \times \Sigma_a \times C_a$. Now, if we set $\mathbf{v} = \mathbf{0}$ in (3.2), and define: $\Delta_{ba} =: \delta_{ba}(\mathbf{0})$; $\mathbf{u}_{ba} =: \Phi_{ba}(\mathbf{0})$, then we find that $\partial T'/\partial T$ and $\partial P'/\partial T$ are constants:

$$\frac{\partial T'}{\partial T} = \Delta_{ba} \quad ; \quad \frac{\partial P'}{\partial P} = \Delta_{ba}\mathbf{u}_{ba}.$$

Finally, since the right-hand sides of (3.2) depend only on $\mathbf{v} \in C_a$, then $\partial T'/\partial P$ and $\partial P'/\partial P$ also are constants. We denote their values by:

$$\tau_{ba} =: \frac{\partial T'}{\partial P} \quad ; \quad \sigma_{ba} =: \frac{\partial P'}{\partial P}.$$

We have just proved that ϕ_{ba} is an affine isomorphism, provided that its differential is the constant linear function:

$$d\phi_{ba} = \begin{pmatrix} \Delta_{ba}\mathbf{e}_0^* \otimes \mathbf{e}_0' & \tau_{ba} \otimes \mathbf{e}_0' \\ \mathbf{e}_0^* \otimes \Delta_{ba}\mathbf{u}_{sr} & \sigma_{ba} \end{pmatrix}.$$

$\square$

The space-time transformation connecting two inertially equivalent reference frames is completely determined by the four quantities:

1. the number $\Delta_{ba} \in \mathbb{R}^+$;
2. the covector $\tau_{ba} \in \mathbf{\Sigma}_a^*$;
3. the vector $\mathbf{u}_{ba} \in \mathbf{\Sigma}_b$;
4. the linear isomorphism $\sigma_{ba} : \mathbf{\Sigma}_a \to \mathbf{\Sigma}_b$;

we call them the *coefficients of the transformation* ϕ_{ba}. Any other kinematical quantity associated with ϕ_{ba} can be expressed as a function of these coefficients. For example, by (3.2) we derive:

$$\begin{aligned} \delta_{ba}(\mathbf{v}) &= \Delta_{ba} + < \tau_{ba} \mid \mathbf{v} > \\ \Phi_{ba}(\mathbf{v}) &= (\Delta_{ba} + < \tau_{ba} \mid \mathbf{v} >)^{-1}(\Delta_{ba}\mathbf{u}_{ba} + \sigma_{ba}\mathbf{v}). \end{aligned} \tag{3.3}$$

The most important features of the inertial equivalence is that it divides the set of all the reference frames of M into equivalence classes and, moreover, in any of these classes suitable *cocycle relations* hold among the coefficients of the space-time transformations. Therefore, the inertia principle, mathematically, consists in singling out an equivalence class of reference frames. Moreover, the cocycle relations can be interpreted as the most general compatibility conditions for the determination of the coefficients of space-time transformations in any theory of space and time . We now determine the cocycle relations.

Theorem 2: *The inertial equivalence is an equivalence relation in the set of all the global space-time reference frames of M. Moreover, if a, b, c are three frames belonging to the same class, then the coefficients of $\phi_{ab}, \phi_{ba}, \phi_{cb}$, and ϕ_{ca}, fulfil the following cocycle relations:*

$$\begin{aligned} (\Delta_{ab} + < \tau_{ab} \mid \mathbf{u}_{ba} >)\Delta_{ba} &= 1 \\ \Delta_{ab}\mathbf{u}_{ab} + \sigma_{ab}\mathbf{u}_{ba} &= 0 \\ \Delta_{ab}\tau_{ba} + \tau_{ab} \circ \sigma_{ba} &= 0 \\ \sigma_{ab} \circ (\sigma_{ba} - \tau_{ba} \otimes \mathbf{u}_{ba}) &= id_{\mathbf{\Sigma}_a} \\ (\Delta_{ba} + < \tau_{ba} \mid \mathbf{u}_{ab} >)\Delta_{ab} &= 1 \\ \Delta_{ba}\mathbf{u}_{ba} + \sigma_{ba}\mathbf{u}_{ab} &= 0 \\ \Delta_{ba}\tau_{ab} + \tau_{ba} \circ \sigma_{ab} &= 0 \\ \sigma_{ba} \circ (\sigma_{ab} - \tau_{ab} \otimes \mathbf{u}_{ab}) &= id_{\mathbf{\Sigma}_b} \end{aligned} \tag{3.4}$$

$$\begin{aligned} (\Delta_{cb} + < \tau_{cb} \mid \mathbf{u}_{ba} >)\Delta_{ba} &= \Delta_{ca} \\ (\Delta_{cb} + < \tau_{cb} \mid \mathbf{u}_{ba} >)^{-1}(\Delta_{cb}\mathbf{u}_{cb} + \sigma_{cb}\mathbf{u}_{ba}) &= \Phi_{cb}(\mathbf{u}_{ba}) = \mathbf{u}_{ca} \\ \Delta_{cb}\tau_{ba} + \tau_{cb} \circ \sigma_{ba} &= \tau_{ca} \\ \sigma_{cb} \circ (\sigma_{ba} - \tau_{ba} \otimes \mathbf{u}_{bc}) &= \sigma_{ca}. \end{aligned} \tag{3.5}$$

Proof. The relation $\sim_I$ is manifestly reflexive, provided that $\phi_{aa} = id_{\Theta_a \times \Sigma_a}$. Let's now assume $a \sim_I b$. Since $\phi_{ab} = \phi_{ba}^{-1}$, then ϕ_{ab} is an affine isomorphism, and $d\phi_{ab} = (d\phi_{ba})^{-1}$; so if we set:

$$d\phi_{ab} = \begin{pmatrix} \Delta_{ab}\mathbf{e}_0'^* \otimes \mathbf{e}_0 & \tau_{ab} \otimes \mathbf{e}_0 \\ \mathbf{e}_0'^* \otimes \Delta_{ab}\mathbf{u}_{ab} & \sigma_{ab} \end{pmatrix},$$

then the identities:

$$d\phi_{ab} \circ d\phi_{ba} = id_{\boldsymbol{\Theta}_a \oplus \boldsymbol{\Sigma}_a} \quad ; \quad d\phi_{ba} \circ d\phi_{ab} = id_{\boldsymbol{\Theta}_b \oplus \boldsymbol{\Sigma}_b},$$

lead to the cocycle relations (3.4). Moreover, for any uniform motion with respect to b, we find:

$$\phi_{ab}(T_0' + t'\mathbf{e}_0', P_0' + t'\mathbf{v}') = \phi_{ab}(T_0', P_0') + d\phi_{ab}(t'\mathbf{e}_0', t'\mathbf{v}') =$$
$$= (T_0 + \delta_{ab}(\mathbf{v}')t'\mathbf{e}_0, P_0 + \delta_{ab}(\mathbf{v}')t'\mathbf{v}),$$

where:

$$(T_0, P_0) =: \phi_{ab}(T_0', P_0'),$$
$$\delta_{ab}(\mathbf{v}') =: \Delta_{ab} + < \tau_{ab} \mid \mathbf{v}' >,$$
$$\mathbf{v} =: \Phi_{ab}(\mathbf{v}') =: (\Delta_{ab} + < \tau_{ab} \mid \mathbf{v}' >)^{-1}(\Delta_{ab}\mathbf{u}_{ab} + \sigma_{ab}\mathbf{v}').$$

Moreover, from (3.4) we derive that:

$$\delta_{ab}(\mathbf{v}') = \frac{1}{\delta_{ba}(\Phi_{ba}^{-1}(\mathbf{v}'))} \quad ; \quad \Phi_{ab}(\mathbf{v}') = \Phi_{ba}^{-1}(\mathbf{v}'); \tag{3.6}$$

hence, the function $\delta_{ab} : C_b \to \mathbb{R}$ is smooth and positive valued, and the function $\Phi_{ab} : C_b \to \boldsymbol{\Sigma}_a$ is an orientation preserving smooth diffeomorphism of C_b onto C_a. This means that: $b \sim_I a$.

Finally, let's assume that $a \sim_I b$ and $b \sim_I c$. Since $\phi_{ca} = \phi_{cb} \circ \phi_{ba}$, then ϕ_{ca} is an affine isomorphism; so if we set:

$$d\phi_{ca} = \begin{pmatrix} \Delta_{ca}\mathbf{e}_0^* \otimes \mathbf{e}_0'' & \tau_{ca} \otimes \mathbf{e}_0'' \\ \mathbf{e}_0' \otimes \Delta_{ca}\mathbf{u}_{ca} & \sigma_{ca} \end{pmatrix},$$

then the chain rule: $d\phi_{ca} = d\phi_{cb} \circ d\phi_{ba}$, leads to the cocycle relations (3.5). Now, for any uniform motion with respect to a, we find:

$$\phi_{ca}(T_0 + t\mathbf{e}_0, P_0 + t\mathbf{v}) = \phi_{ca}(T_0, P_0) + d\phi_{ca}(t\mathbf{e}_0, t\mathbf{v}) =$$
$$= (T_0'' + \delta_{ca}(\mathbf{v})t\mathbf{e}_0'', P_0'' + \delta_{ca}(\mathbf{v})t\mathbf{v}''),$$

where:

$$(T_0'', P_0'') =: \phi_{ca}(T_0, P_0),$$
$$\delta_{ca}(\mathbf{v}) =: \Delta_{ca} + < \tau_{ca} \mid \mathbf{v}) >,$$
$$\mathbf{v}'' = \Phi_{ca}(\mathbf{v}) =: (\Delta_{ca} + < \tau_{ca} \mid \mathbf{v}) >)^{-1}(\Delta_{ca}\mathbf{u}_{ca} + \sigma_{ca}\mathbf{v}).$$

Moreover, from (3.5) we derive that:

$$\delta_{ca}(\mathbf{v}) = \delta_{cb}(\Phi_{ba}(\mathbf{v})) \cdot \delta_{ba}(\mathbf{v}) \quad ; \quad \Phi_{ca}(\mathbf{v}) = \Phi_{cb} \circ \Phi_{ba}(\mathbf{v}); \tag{3.7}$$

hence, the function $\delta_{ca} : C_a \to \mathbb{R}$ is smooth and positive definite, and the function $\Phi_{ca} : C_a \to \mathbf{\Sigma}_c$ is an orientation preserving C^∞-diffeomorphism of C_a onto C_c. This means that $a \sim_I c$. □

4. Kinematical meaning of the coefficients of a transformation.

Let a and b be two space-time reference frames of M which are inertially equivalent. The coefficients of the space-time transformation ϕ_{ba} have the kinematical meanings of dragging velocity, deformation ratios of time and space measures, and simultaneity defect, affecting ϕ_{ba}. To state this more clearly, we fix an event, $o \in M$, in order to single out origins in time and space within a and b, by:

$$\phi_a(o) = (T_0, O) \quad ; \quad \phi_b(o) = (T_0', O').$$

Then, any event $x \in M$ is represented by its time and space coordinates by:

$$\begin{aligned} \phi_a(x) &= (T, P) = (T_0, O) + (t\mathbf{e}_0, \mathbf{r}) \\ \phi_b(x) &= (T', P') = (T_0', O') + (t'\mathbf{e}_0', \mathbf{r}'), \end{aligned}$$

and the space-time transformation, $(T', P') = \phi_{ba}(T, P)$, is represented by the linear transformation:

$$\begin{aligned} t' &= \Delta_{ba} t + < \tau_{ba} \mid \mathbf{r} > \\ \mathbf{r}' &= \Delta_{ba} t \mathbf{u}_{ba} + \sigma_{ba} \mathbf{r}. \end{aligned} \tag{4.1}$$

The vector $\mathbf{u}_{ba}$ represents the *dragging velocity* of the space Σ_a with respect to b. In other words, $\mathbf{u}_{ba} = \Phi_{ba}(\mathbf{0}) \in C_b$ is the velocity, measured in b, of any particle which is at rest in a. The dragging velocity $\mathbf{u}_{ba}$ is related to the dragging velocity $\mathbf{u}_{ab}$ of the space Σ_b with respect to a by the cocycle relation $(3.4)_6$:

$$\mathbf{u}_{ba} = -\Delta_{ba}^{-1} \sigma_{ba} \mathbf{u}_{ab}.$$

Therefore, the equations (4.1) may be also expressed as follows:

$$\begin{aligned} t' &= \Delta_{ba} t + < \tau_{ba} \mid \mathbf{r} > \\ \mathbf{r}' &= \sigma_{ba}(\mathbf{r} - t\mathbf{u}_{ab}), \end{aligned} \tag{4.2}$$

while the transformation rule for velocities takes the form:

$$\mathbf{v}' = \Phi_{ba}(\mathbf{v}) = (\Delta_{ba} + < \tau_{ba} \mid \mathbf{v} >)^{-1} \sigma_{ba}(\mathbf{v} - \mathbf{u}_{ab}). \tag{4.3}$$

Finally, the transformation rules for velocity and acceleration of arbitrary motions can be expressed by means of the coefficients of ϕ_{ba}. Indeed, let $m(t) = (T_0, O) + (t\mathbf{e}_0, \mathbf{r}(t))$ be any

motion referred to a. The same motion, referred to b, $m'(t') = (T'_0, O') + (t'\mathbf{e}'_0, \mathbf{r}'(t'))$, is expressed as a function of the time scale of a by means of:

$$\phi_{ba} \circ m(t) = m'(f(t)),$$

where:

$$f(t) = \Delta_{ba} t + < \tau_{ba} \mid \mathbf{r}(t) > .$$

Now, performing the time derivatives, we find the transformation rules:

$$\begin{aligned} \mathbf{v}'(t') \mid_{t'=f(t)} &= (\Delta_{ba} + < \tau_{ba} \mid \mathbf{v}(t) >)^{-1} \sigma_{ba}(\mathbf{v}(t) - \mathbf{u}_{ab}) = \Phi_{ba}(\mathbf{v}(t)) \\ \mathbf{a}'(t') \mid_{t'=f(t)} &= (\Delta_{ba} + < \tau_{ba} \mid \mathbf{v}(t) >)^{-2} (\sigma_{ba}\mathbf{a}(t) - < \tau_{ba} \mid \mathbf{a}(t) > \Phi_{ba}(\mathbf{v}(t))). \end{aligned} \tag{4.4}$$

Accordingly to the inertia principle, from (4.4) it follows that the uniform motion of a particle has an "absolute meaning". Nevertheless, the acceleration vector itself is generally frame-dependent.

Let now $x_1, x_2 \in M$ be two events which are genidentic with respect to a (cfr. sect. 2). Then x_1 and x_2 are connected by the propagation of a particle which is at rest in a at the point $\mathbf{r}_2 = \mathbf{r}_1$, hence:

$$\frac{t'_2 - t'_1}{t_2 - t_1} = \delta_{ba}(\mathbf{o}) = \Delta_{ba},$$

where $t'_2 - t'_1$ and $t_2 - t_1$ measure the time separations between x_1 and x_2 respectively in b and in a. Hence, the coefficient $\Delta_{ba} \in \mathbb{R}^+$ represents the *deformation ratio of time measures*, between a-genidentic events, affecting the transformation ϕ_{ba}.

Let's now assume that x_1 and x_2 are simultaneous with respect to a. Then $t_2 = t_1$ and, by (4.2), it follows that:

$$t'_2 - t'_1 = < \tau_{ba} \mid \mathbf{r}_2 - \mathbf{r}_1 > .$$

Hence, the number $< \tau_{ba} \mid \mathbf{r}_2 - \mathbf{r}_1 >$ measures the time separation, with respect to b, between x_1 and x_2. Therefore, the coefficient $\tau_{ba} \in \mathbf{\Sigma}^*_a$ represents the *simultaneity defect* affecting the transformation ϕ_{ba}. The simultaneity of x_1 and x_2 is conserved by ϕ_{ba} if and only if: $\mathbf{r}_2 - \mathbf{r}_1 \in \ker\, \tau_{ba}$; therefore we call $\ker\, \tau_{ba} \subseteq \mathbf{\Sigma}_a$ the *subspace of conserved simultaneity* associated to ϕ_{ba}.

Now, we consider a rod which is at rest in b, and denote by $\mathbf{r}' \in \mathbf{\Sigma}_b$ the vector connecting its extreme points. Since these points are uniformly moving in a with dragging velocity $\mathbf{u}_{ab}$, then we derive from (4.2) that their positions in a, at any time, are connected by the vector $\mathbf{r} \in \mathbf{\Sigma}_a$ determined by: $\mathbf{r}' = \sigma_{ba}\mathbf{r}$. Hence, for any rod which is at rest in b, σ_{ba} maps its instantaneous configurations in a onto its configuration in b. We call σ_{ba} the *space transformation* associated to ϕ_{ba}. From σ_{ba} we derive the function:

$$\lambda_{ba}(\mathbf{r}) =: \frac{\mid \mathbf{r}' \mid}{\mid \mathbf{r} \mid} = \frac{h_b(\sigma_{ba}\mathbf{r}, \sigma_{ba}\mathbf{r})^{1/2}}{h_a(\mathbf{r}, \mathbf{r})^{1/2}}, \tag{4.5}$$

which represents the *deformation ratio of space measures* affecting the transformation ϕ_{ba}. Moreover, we define the *principal deformation ratios of* ϕ_{ba} as the square roots of the eigenvalues of the symmetric bilinear form $h_b \circ (\sigma_{ba} \times \sigma_{ba})$, with respect to h_a. Finally, we call *principal vector of* ϕ_{ba}, any eigenvector of $h_b \circ (\sigma_{ba} \times \sigma_{ba})$. Clearly, a non zero vector, $\mathbf{r} \in \mathbf{\Sigma}_a$, is principal if and only if:

$$\forall \boldsymbol{\xi} \in \mathbf{\Sigma}_a \quad , \quad h_b(\sigma_{ba}\boldsymbol{\xi}, \sigma_{ba}\mathbf{r}) = \lambda^2_{ba}(\mathbf{r}) h_a(\boldsymbol{\xi}, \mathbf{r}).$$

We call *principal basis* of the transformation ϕ_{ba}, any oriented orthonormal basis of $\mathbf{\Sigma}_a$, consisting of principal vectors.

Theorem 3: *If* $\{\mathbf{e}_\alpha, \alpha = 1,2,3\}$*, is a principal basis of* ϕ_{ba}*, then the vectors:*

$$\mathbf{e}'_\alpha =: \lambda_{ba}(\mathbf{e}_\alpha)^{-1} \sigma_{ba} \mathbf{e}_\alpha \quad , \quad \alpha = 1,2,3, \tag{4.6}$$

form an oriented, orthonormal basis in $\mathbf{\Sigma}_b$*.*

Proof. Since σ_{ba} preserves the orientations, and $\lambda_{ba}(\mathbf{e}_\alpha) > 0$, then $\{\mathbf{e}'_\alpha, \alpha = 1,2,3\}$, is an oriented basis of $\mathbf{\Sigma}_b$. Moreover, since the $\mathbf{e}_\alpha$'s are principal vectors, then:

$$\begin{aligned} h_b(\mathbf{e}'_\alpha, \mathbf{e}'_\beta) &= \lambda_{ba}(\mathbf{e}_\alpha)^{-1} \lambda_{ba}(\mathbf{e}_\beta)^{-1} h_b(\sigma_{ba}\mathbf{e}_\alpha, \sigma_{ba}\mathbf{e}_\beta) = \\ &= \lambda_{ba}(\mathbf{e}_\alpha) \lambda_{ba}(\mathbf{e}_\beta)^{-1} h_a(\mathbf{e}_\alpha, \mathbf{e}_\beta) = \\ &= \delta_{\alpha\beta}. \end{aligned}$$

Hence $\{\mathbf{e}'_\alpha, \alpha = 1,2,3\}$ is orthonormal. □

Clearly, the same kinematical meanings are found for the coefficients Δ_{ab}, τ_{ab} and σ_{ab} of the inverse transformation ϕ_{ab}. Moreover, they are correlated to Δ_{ba}, τ_{ba} and σ_{ba} by the cocycle relations (3.4). These cocycle relations also lead to the following identities:

$$\begin{gathered} \sigma_{ba}(\ker\, \tau_{ba}) = \ker\, \tau_{ab} \\ \forall \boldsymbol{\xi} \in \ker\, \tau_{ba} \ , \ \sigma_{ab} \circ \sigma_{ba} \boldsymbol{\xi} = \boldsymbol{\xi} \\ \forall \boldsymbol{\xi} \in \ker\, \tau_{ba} \ , \ \lambda_{ab}(\sigma_{ba}\boldsymbol{\xi}) = \lambda_{ba}(\boldsymbol{\xi})^{-1} \\ \sigma_{ab}(\ker\, \tau_{ab}) = \ker\, \tau_{ba} \\ \forall \boldsymbol{\xi}' \in \ker\, \tau_{ab} \ , \ \sigma_{ba} \circ \sigma_{ab} \boldsymbol{\xi}' = \boldsymbol{\xi}' \\ \forall \boldsymbol{\xi}' \in \ker\, \tau_{ab} \ , \ \lambda_{ab}(\sigma_{ab}\boldsymbol{\xi}') = \lambda_{ab}(\boldsymbol{\xi}')^{-1}. \end{gathered} \tag{4.7}$$

Indeed, by $(3.4)_3$ and $(3.4)_7$ it follows that, for any $\boldsymbol{\xi} \in \ker\, \tau_{ba}$, we have:

$$< \tau_{ab} \mid \sigma_{ba}\boldsymbol{\xi} >= -\Delta_{ab} < \tau_{ba} \mid \boldsymbol{\xi} >= 0,$$

hence: $\sigma_{ba}(\ker\ \tau_{ba}) \subseteq \ker\ \tau_{ab}$; then, the identity $(4.7)_1$ follows from: $\dim\ker\ \tau_{ab} = \dim\ker\ \tau_{ba}$. Moreover, by $(3.4)_4$ it follows that any $\boldsymbol{\xi} \in \ker\ \tau_{ba}$ also fulfils the equations:

$$\begin{aligned} \boldsymbol{\xi} &= \sigma_{ab} \circ (\sigma_{ba} - \tau_{ba} \otimes \mathbf{u}_{ba})\boldsymbol{\xi} = \\ &= \sigma_{ab}\sigma_{ba}\boldsymbol{\xi} - < \tau_{ba} \mid \boldsymbol{\xi} > \sigma_{ab}\mathbf{u}_{ba} = \\ &= \sigma_{ab}\sigma_{ba}\boldsymbol{\xi}, \end{aligned}$$

and:

$$\begin{aligned} \lambda_{ab}(\sigma_{ba}\boldsymbol{\xi}) &= \frac{h_a(\sigma_{ab}\sigma_{ba}\boldsymbol{\xi}, \sigma_{ab}\sigma_{ba}\boldsymbol{\xi})^{1/2}}{h_b(\sigma_{ba}\boldsymbol{\xi}, \sigma_{ba}\boldsymbol{\xi})^{1/2}} = \\ &= \frac{h_a(\boldsymbol{\xi}, \boldsymbol{\xi})^{1/2}}{h_b(\sigma_{ba}\boldsymbol{\xi}, \sigma_{ba}\boldsymbol{\xi})^{1/2}} = \\ &= \frac{1}{\lambda_{ba}(\boldsymbol{\xi})}. \end{aligned}$$

The proof of identities $(4.7)_{4,5,6}$ is similar.

Let's now refer the space $\boldsymbol{\Sigma}_a$ to a principal basis of ϕ_{ba}, $\{\mathbf{e}_\alpha, \alpha = 1,2,3\}$, and refer the space $\boldsymbol{\Sigma}_b$ to the oriented, orthonormal basis defined by (4.6). By introducing the components of τ_{ba} and $\mathbf{u}_{ab}$:

$$\tau_\alpha =:< \tau_{ba} \mid \mathbf{e}_\alpha > \quad ; \quad u^\alpha =: h_a(\mathbf{u}_{ab}, \mathbf{e}_\alpha) \quad , \quad \alpha = 1,2,3,$$

we derive, from (4.2), the following coordinate expression of the space-time transformation ϕ_{ba}:

$$\begin{aligned} t' &= \Delta_{ba} t + \tau_1 x^1 + \tau_2 x^2 + \tau_3 x^3 \\ x'^\alpha &= \lambda_{ba}(\mathbf{e}_\alpha)(x^\alpha - u^\alpha t) \qquad \alpha = 1,2,3. \end{aligned} \tag{4.8}$$

Clearly, the inverse transformation formulas have not the same simple structure of (4.8). This is due to the fact that, although the vectors $\mathbf{e}'_\alpha = \lambda_{ba}(\mathbf{e}_\alpha)^{-1}\sigma_{ba}\mathbf{e}_\alpha$, $\alpha = 1,2,3$ form an oriented and orthonormal basis, they are not principal vectors of the inverse transformation ϕ_{ab}, in general.

5. The reciprocity conditions for a transformation.

Formulas (4.8) are the simplest expression, in terms of coordinates, of the transformation ϕ_{ba}. In fact, no feature can be find in the cocycle relations (3.4) - i.e. in the inertial equivalence relation - leading to a simpler expression. Such a simplification only follows from some further assumptions correlating the space transformation, σ_{ba}, and the simultaneity defect, τ_{ba}, to the dragging velocity, $\mathbf{u}_{ab}$, by means of the euclidean metrics of Σ_a and Σ_b. These assumptions, which we call the *reciprocity conditions*, are:

1. *The dragging velocity,* $\mathbf{u}_{ab}$*, is a principal vector of* ϕ_{ba}*.*
2. *The deformation ratio of space measures,* λ_{ba}*, is constant on the orthogonal plane,* $(\mathbf{u}_{ab})^\perp$*.*

3. *The simultaneity defect, τ_{ba}, depends linearly on the dragging velocity:*

$$\forall \boldsymbol{\xi} \in \boldsymbol{\Sigma}_a \quad , \quad < \tau_{ba} \mid \boldsymbol{\xi} >= \theta h_a(\mathbf{u}_{ab}, \boldsymbol{\xi}),$$

where $\theta \in \mathbb{R}$.

Although not a consequence of the inertial equivalence relation, the reciprocity conditions are taken in any theory of space and time. In fact, in a relativistic theory, the reciprocity conditions are taken to describe all the space-time transformations, while in a theory founded on the absolute frame, they are taken to describe only the space-time transformations from the absolute frame to any other "relative" frame.

The conditions 1 and 2 concern the structure of the spatial deformations affecting the transformation ϕ_{ba}. More precisely, the condition 1 says that the ratio:

$$\lambda =: \lambda_{ba}(\mathbf{u}_{ab}) = \Delta_{ba} \frac{u_{ba}}{u_{ab}},$$

(where $u_{ba} =: \mid \mathbf{u}_{ba} \mid$, $u_{ab} =: \mid \mathbf{u}_{ab} \mid$), is a principal deformation ratio of ϕ_{ba}. Condition 1 have the following kinematical meaning: any rod at rest in b, which is orthogonal to the dragging velocity $\mathbf{u}_{ba}$, have instantaneous configurations, in a, which are orthogonal to the dragging velocity $\mathbf{u}_{ab}$.

Theorem 4: *The condition 1 is equivalent to:*

$$\sigma_{ba}(\mathbf{u}_{ab})^{\perp} = (\mathbf{u}_{ba})^{\perp}. \tag{5.1}$$

Proof. If $\mathbf{u}_{ab}$ is assumed to be principal, then any $\boldsymbol{\xi} \in (\mathbf{u}_{ab})^{\perp}$ fulfils:

$$\begin{aligned} h_b(\sigma_{ba}\boldsymbol{\xi}, \mathbf{u}_{ba}) &= -\Delta_{ba}^{-1} h_b(\sigma_{ba}\boldsymbol{\xi}, \sigma_{ba}\mathbf{u}_{ab}) = \\ &= -\Delta_{ba}^{-1} \lambda^2 h_a(\boldsymbol{\xi}, \mathbf{u}_{ab}) = \\ &= 0, \end{aligned}$$

hence: $\sigma_{ba}(\mathbf{u}_{ab})^{\perp} \subseteq (\mathbf{u}_{ba})^{\perp}$. Therefore, since dim $\sigma_{ba}(\mathbf{u}_{ab})^{\perp} = 2 =$dim $(\mathbf{u}_{ba})^{\perp}$, then we find (5.1). Conversely, if the equation (5.1) is assumed to hold, then by decomposing any $\boldsymbol{\xi} \in \boldsymbol{\Sigma}_a$ as:

$$\boldsymbol{\xi} = c\mathbf{u}_{ab} + \boldsymbol{\xi}^{\perp} \quad , \quad \boldsymbol{\xi}^{\perp} \in (\mathbf{u}_{ab})^{\perp},$$

we find that:

$$\begin{aligned} h_b(\sigma_{ba}\boldsymbol{\xi}, \sigma_{ba}\mathbf{u}_{ab}) &= c h_b(\sigma_{ba}\mathbf{u}_{ab}, \sigma_{ba}\mathbf{u}_{ab}) + h_b(\sigma_{ba}\boldsymbol{\xi}^{\perp}, \sigma_{ba}\mathbf{u}_{ab}) = \\ &= c\lambda^2 h_a(\mathbf{u}_{ab}, \mathbf{u}_{ab}) - \Delta_{ba} h_b(\sigma_{ba}\boldsymbol{\xi}^{\perp}, \mathbf{u}_{ba}) = \\ &= \lambda^2 h_a(c\mathbf{u}_{ab}, \mathbf{u}_{ab}) = \\ &= \lambda^2 h_a(c\mathbf{u}_{ab} + \boldsymbol{\xi}^{\perp}, \mathbf{u}_{ab}) = \\ &= \lambda^2 h_a(\boldsymbol{\xi}, \mathbf{u}_{ab}), \end{aligned}$$

i.e. $\mathbf{u}_{ab}$ is a principal vector of ϕ_{ba}. □

Condition 2 means that the rods at rest in b whose instantaneous configurations in a are orthogonal to the dragging velocity $\mathbf{u}_{ab}$, all define the same deformation ratio of space measures. Therefore, no direction in the plane $(\mathbf{u}_{ab})^{\perp}$ is *privileged* by the space-time transformation. Condition 2, together with condition 1, implies that this constant ratio:

$$\mu =: \lambda_{ba}(\boldsymbol{\xi}) \quad , \quad \boldsymbol{\xi} \in (\mathbf{u}_{ab})^{\perp},$$

is a principal deformation ratio and, then, that the orthogonal plane, $(\mathbf{u}_{ab})^{\perp}$, is a *principal subspace* of ϕ_{ba}. A further kinematical meaning of condition 2 is founded in the following theorem.

Theorem 5: *If* $\mathbf{u}_{ab}$ *is a principal vector, then condition 2 is valid if and only if the restricted space transformation:*

$$\sigma_{ba} : (\mathbf{u}_{ab})^{\perp} \to (\mathbf{u}_{ba})^{\perp}$$

is a conformal mapping.

Proof. Clearly, the restricted mapping is conformal if and only if the following identity holds:

$$h_b(\sigma_{ba}\boldsymbol{\xi}, \sigma_{ba}\boldsymbol{\eta}) = \lambda_{ba}(\boldsymbol{\xi})\lambda_{ba}(\boldsymbol{\eta})h_a(\boldsymbol{\xi}, \boldsymbol{\eta}), \tag{5.2}$$

for any couple of nonzero vectors, $\boldsymbol{\xi}, \boldsymbol{\eta} \in (\mathbf{u}_{ab})^{\perp}$. Now, if condition 2 is true, then $(\mathbf{u}_{ab})^{\perp}$ is a principal subspace, and λ_{ba} is constant on it; hence:

$$\begin{aligned} h_b(\sigma_{ba}\boldsymbol{\xi}, \sigma_{ba}\boldsymbol{\eta}) &= \mu^2 h_a(\boldsymbol{\xi}, \boldsymbol{\eta}) = \\ &= \lambda_{ba}(\boldsymbol{\xi})\lambda_{ba}(\boldsymbol{\eta})h_a(\boldsymbol{\xi}, \boldsymbol{\eta}). \end{aligned}$$

Conversely, let's assume that (5.2) holds - together with the condition 1. Now, if $\boldsymbol{\eta} \in (\mathbf{u}_{ab})^{\perp}$ is any principal vector of ϕ_{ba}, then any nonzero vector $\boldsymbol{\xi} \in (\mathbf{u}_{ab})^{\perp}$ obeys the relation:

$$h_b(\sigma_{ba}\boldsymbol{\xi}, \sigma_{ba}\boldsymbol{\eta}) = \lambda_{ba}(\boldsymbol{\eta})^2 h_a(\boldsymbol{\xi}, \boldsymbol{\eta}).$$

Moreover, if $\boldsymbol{\xi}$ isn't orthogonal to $\boldsymbol{\eta}$, then by (5.2) it follows that $\sigma_{ba}\boldsymbol{\xi} \in (\mathbf{u}_{ba})^{\perp}$ isn't orthogonal to $\sigma_{ba}\boldsymbol{\eta}$, and that:

$$\lambda_{ba}(\boldsymbol{\xi}) = \lambda_{ba}(\boldsymbol{\eta}).$$

From this we infer that λ_{ba} is constant on $(\mathbf{u}_{ab})^{\perp}$. □

Conditions 1 and 2 imply that any oriented orthonormal basis of $\boldsymbol{\Sigma}_a$ of the type:

$$\mathbf{e}_1 =: u_{ab}^{-1}\mathbf{u}_{ab} \quad ; \quad \mathbf{e}_{\alpha} \in (\mathbf{u}_{ab})^{\perp} \quad , \quad \alpha = 2, 3, \tag{5.3}$$

is a principal basis of ϕ_{ba}. Moreover, the oriented orthonormal basis of $\mathbf{\Sigma}_b$ defined in (4.6) is given by:

$$\mathbf{e}'_1 = \lambda^{-1}\sigma_{ba}\mathbf{e}_1 \quad ; \quad \mathbf{e}'_\alpha = \mu^{-1}\sigma_{ba}\mathbf{e}_\alpha \quad , \quad \alpha = 2, 3,$$

and fulfils:

$$\mathbf{e}'_1 = -u_{ba}^{-1}\mathbf{u}_{ba} \quad ; \quad \mathbf{e}'_\alpha \in (\mathbf{u}_{ba})^\perp \quad , \quad \alpha = 2, 3. \tag{5.4}$$

Now, referred to the bases (5.3) and (5.4), the coordinate expression of the space-time transformation ϕ_{ba} reduces to the simpler form:

$$\begin{aligned} t' &= \Delta_{ba}t + \tau_1 x^1 + \tau_2 x^2 + \tau_3 x^3 \\ x'^1 &= \lambda(x^1 - u_{ab}t) \\ x'^\alpha &= \mu x^\alpha \quad , \quad \alpha = 2, 3. \end{aligned}$$

The condition 3 concerns the simultaneity defect affecting the space-time transformation ϕ_{ba}. It means that simultaneity be conserved, in the transformation ϕ_{ba}, for any couple of events which are spatially separated, in a, by a vector which is orthogonal to the dragging velocity, $\mathbf{u}_{ab}$. Indeed, if the events $x_1, x_2 \in M$ are simultaneous with respect to a, then their time separation, measured in b, is:

$$t'_2 - t'_1 =< \tau_{ba} \mid \mathbf{r}_2 - \mathbf{r}_1 >= \theta h_a(\mathbf{u}_{ab}, \mathbf{r}_2 - \mathbf{r}_1).$$

Hence, if $\mathbf{r}_2 - \mathbf{r}_1 \in (\mathbf{u}_{ab})^\perp$, then: $t'_2 = t'_1$.

The condition 3 is manifestly equivalent to the inclusion:

$$(\mathbf{u}_{ab})^\perp \subseteq \ker\, \tau_{ba}. \tag{5.5}$$

Clearly, inclusion (5.5) becames an identity, if $\tau_{ba} \neq 0$. Moreover, in any basis of the type (5.3), the components of τ_{ba} are:

$$\tau_1 = u_{ab}\theta \quad ; \quad \tau_2 = \tau_3 = 0.$$

We now suppose that all the reciprocity conditions hold for the space-time transformation ϕ_{ba}. Then the principal bases and the subspace of conserved simultaneity of ϕ_{ba} depend only on the dragging velocity, $\mathbf{u}_{ab}$, by means of (5.3) and (5.5). Moreover, the coordinate expression of ϕ_{ba}, referred to the bases (5.3) and (5.4), reduces to the simplified form:

$$\begin{aligned} t' &= \Delta_{ba}t + u_{ab}\theta x^1 \\ x'^1 &= \lambda(x^1 - u_{ab}t) \\ x'^\alpha &= \mu x^\alpha \quad , \quad \alpha = 2, 3. \end{aligned} \tag{5.6}$$

Formulas (5.6) are the simplest expression in terms of coordinates of the space-time transformation ϕ_{ba}, as a consequence of the reciprocity conditions. We call them the *special coordinate expression* of ϕ_{ba}. Formulas (5.6) shows that the transformation ϕ_{ba} is completely determined by the dragging velocity, $\mathbf{u}_{ab}$, and the four parameters:

$$\Delta_{ba}, \lambda, \mu \in \mathbb{R}^{+} \quad ; \quad \theta \in \mathbb{R}. \tag{5.7}$$

Moreover, it is a simple matter to show that, if a space-time transformation ϕ_{ba} has a special coordinate expression, then its coefficients must satisfy the reciprocity conditions.

An important feature of the reciprocity conditions is that they are symmetric under the exchange of the two reference frames involved in the space-time transformation.

Theorem 6: *The reciprocity conditions 1-3 are fulfilled by the coefficients of the inverse transformation, ϕ_{ab}, if and only if they are fulfilled by the coefficients of ϕ_{ba}.*

Proof. It is sufficient to restrict our proof to the "if" part of the theorem. Let's assume that ϕ_{ba} fulfils the reciprocity conditions. To see that the inverse transformation, ϕ_{ab}, also fulfils these conditions, we must show that the coefficients of ϕ_{ab} fulfil the equations:

$$\sigma_{ab}(\mathbf{u}_{ba})^{\perp} = (\mathbf{u}_{ab})^{\perp} \tag*{(5.1)$'$}$$

$$\forall \boldsymbol{\xi}', \boldsymbol{\eta}' \in (\mathbf{u}_{ba})^{\perp} \quad , \quad h_a(\sigma_{ab}\boldsymbol{\xi}', \sigma_{ab}\boldsymbol{\eta}') = \lambda_{ab}(\boldsymbol{\xi}')\lambda_{ab}(\boldsymbol{\eta}')h_b(\boldsymbol{\xi}', \boldsymbol{\eta}') \tag*{(5.2)$'$}$$

$$(\mathbf{u}_{ba})^{\perp} \subseteq \ker\, \tau_{ab}, \tag*{(5.5)$'$}$$

Indeed, from (5.5) and $(4.7)_2$ we find that:

$$\sigma_{ab} \circ \sigma_{ba} \mid_{(\mathbf{u}_{ab})^{\perp}} = \mathrm{id};$$

hence, by (5.1) it follows that:

$$\sigma_{ab}(\mathbf{u}_{ba})^{\perp} = \sigma_{ab} \circ \sigma_{ba}(\mathbf{u}_{ab})^{\perp} = (\mathbf{u}_{ab})^{\perp}.$$

Moreover, by (5.1) and $(4.7)_2$, it follows that any pair of nonzero vectors, $\boldsymbol{\xi}', \boldsymbol{\eta}' \in (\mathbf{u}_{ba})^{\perp}$, fulfil:

$$\begin{aligned} h_a(\sigma_{ab}\boldsymbol{\xi}', \sigma_{ab}\boldsymbol{\eta}') &= h_a(\sigma_{ab}\sigma_{ba}\boldsymbol{\xi}, \sigma_{ab}\sigma_{ba}\boldsymbol{\eta}) = \\ &= h_a(\boldsymbol{\xi}, \boldsymbol{\eta}), \end{aligned}$$

where $\boldsymbol{\xi}' = \sigma_{ba}\boldsymbol{\xi}$, $\boldsymbol{\eta}' = \sigma_{ba}\boldsymbol{\eta}$. Now, by $(4.7)_3$ and (5.2) we find:

$$\begin{aligned} h_a(\sigma_{ab}\boldsymbol{\xi}', \sigma_{ab}\boldsymbol{\eta}') &= h_a(\boldsymbol{\xi}, \boldsymbol{\eta}) = \\ &= \lambda_{ba}(\boldsymbol{\xi})^{-1}\lambda_{ba}(\boldsymbol{\eta})^{-1}h_b(\sigma_{ba}\boldsymbol{\xi}, \sigma_{ba}\boldsymbol{\eta}) = \\ &= \lambda_{ab}(\sigma_{ba}\boldsymbol{\xi})\lambda_{ab}(\sigma_{ba}\boldsymbol{\eta})h_b(\sigma_{ba}\boldsymbol{\xi}, \sigma_{ba}\boldsymbol{\eta}) = \\ &= \lambda_{ab}(\boldsymbol{\xi}')\lambda_{ab}(\boldsymbol{\eta}')h_b(\boldsymbol{\xi}', \boldsymbol{\eta}'). \end{aligned}$$

Finally, by (5.1), (5.5) and $(4.7)_1$ we find:

$$(\mathbf{u}_{ba})^{\perp} = \sigma_{ba}(\mathbf{u}_{ab})^{\perp} \subseteq \sigma_{ba}(\ker\, \tau_{ba}) = \ker\, \tau_{ab}.$$

□

Clearly, the inverse of any special coordinate expression of the transformation ϕ_{ba} is a special coordinate expression of the inverse transformation, ϕ_{ab}. Indeed, the principal deformation ratios affecting ϕ_{ab} are:

$$\lambda' = \lambda_{ab}(\mathbf{u}_{ba}) = \Delta_{ab}\frac{u_{ab}}{u_{ba}} \quad ; \quad \mu' = \frac{1}{\mu};$$

in fact, for any $\boldsymbol{\xi}' \in (\mathbf{u}_{ba})^{\perp}$, we find: $\sigma_{ab}\boldsymbol{\xi}' \in (\mathbf{u}_{ab})^{\perp}$, and then:

$$\mu' = \lambda_{ab}(\boldsymbol{\xi}') = \lambda_{ab}(\sigma_{ba}\sigma_{ab}\boldsymbol{\xi}') = \frac{1}{\lambda_{ba}(\sigma_{ab}\boldsymbol{\xi}')} = \frac{1}{\mu}.$$

Moreover, the basis (5.4) is a principal basis of ϕ_{ab}, and its associated basis in $\boldsymbol{\Sigma}_a$ by means of (4.6) is nothing but the basis (5.3):

$$\lambda'^{-1}\sigma_{ab}\mathbf{e}'_1 = \mathbf{e}_1 \quad ; \quad \mu'^{-1}\sigma_{ab}\mathbf{e}'_\alpha = \mathbf{e}_\alpha \quad , \quad \alpha = 2, 3.$$

Therefore, the coordinate expression of ϕ_{ab}, referred to the bases (5.4) and (5.3), consists of the inverse of (5.6). Since:

$$\tau'_1 =< \tau_{ab} \mid \mathbf{e}'_1 >= -u_{ba}\theta' \quad ; \quad \tau'_\alpha =< \tau_{ab} \mid \mathbf{e}'_\alpha >= 0 \quad , \quad \alpha = 2, 3,$$

then this coordinate expression is:

$$\begin{aligned} t &= \Delta_{ab}t' - u_{ba}\theta' x'^1 \\ x^1 &= \lambda'(x'^1 + u_{ba}t') \\ x^\alpha &= \mu' x'^\alpha \quad , \quad \alpha = 2, 3. \end{aligned} \tag{5.8}$$

Finally, the dragging velocity u_{ba} and the parameters $\Delta_{ab}, \lambda', \mu', \theta'$, can be expressed as functions of $u_{ab}, \Delta_{ba}, \lambda, \mu$ and θ, by means of:

$$u_{ba} = \frac{\lambda}{\Delta_{ba}}u_{ab};$$
$$\Delta_{ab} = \frac{1}{\Delta_{ba} + u_{ab}^2\theta} \quad ; \quad \lambda' = \frac{\Delta_{ba}}{\lambda(\Delta_{ba} + u_{ab}^2\theta)} \quad ; \quad \mu' = \frac{1}{\mu} \quad ; \quad \theta' = \frac{\Delta_{ba}\theta}{\lambda^2(\Delta_{ba} + u_{ab}^2\theta)}.$$

6. Theories of space and time.

As we have remarked in section 3, the inertia principle, mathematically, consists of the singling out of a class of inertially equivalent reference frames, I. This class contains all the reference frames in which the free material particles are uniformly moving.

The aim of the present section is to suggest a definition of what can be meant by a theory of space and time compatible with the inertia principle. To this end, beside the class I, a further mathematical concept must be introduced to determine the space-time transformations between the inertial frames of reference. This object, which we call the *rest frame map family*, consists of a family of mappings:

$$\mathcal{R} =: \{\mathcal{R}_a : C_a \to I / a \in I\},$$

with the following properties:

1. *Any* $\mathcal{R}_a : C_a \to I$ *is one-to-one*;
2. $\forall a, b \in I, \quad \mathcal{R}_a^{-1}(b) = \mathbf{u}_{ab}$;
3. $\forall a, b \in I, \quad \mathcal{R}_b^{-1} \circ \mathcal{R}_a = \Phi_{ba}$.

Properties 1 and 2 mean that if an inertial frame, $a \in I$, is fixed, then any other frame of the class I can be found, in a, as the rest frame associated to some free material particle. Moreover, since the mapping $\mathcal{R}_a$ uniquely determines any $b \in I$ as a function of the dragging velocity $\mathbf{u}_{ab}$, then the space-time transformation ϕ_{ba} is determined as a function of the dragging velocity. Therefore, the coefficients of ϕ_{ba} are found, by means of $\mathcal{R}_a$, as "constitutive" functions of the dragging velocity. To state this more precisely, let's denote by $L_{ab} \subset \mathbf{\Sigma}_a^* \otimes \mathbf{\Sigma}_b$ the set of orientation preserving linear isomorphisms from $\mathbf{\Sigma}_a$ to $\mathbf{\Sigma}_b$ and costruct, for a fixed $a \in I$, the bundle of orientation preserving linear isomorphisms over I by means of:

$$L_a =: \cup_{b \in I} L_{ab}$$

with the bundle projection, $\pi_a : L_a \to I$, defined by: $\pi_a(\sigma) = b \Leftrightarrow: \sigma \in L_{ab}$. Now, the rest frame mapping $\mathcal{R}_a : C_a \to I$ uniquely determines the *constitutive functions*:

$$\Delta^{(a)} : C_a \to \mathrm{I\!R}^+ \quad ; \quad \tau^{(a)} : C_a \to \mathbf{\Sigma}_a^* \quad ; \quad \sigma^{(a)} : C_a \to L_a, \tag{6.1}$$

by means of the equation:

$$d\phi_{ba} =: \begin{pmatrix} \Delta^{(a)}(\mathbf{u})\mathbf{e}_0^* \otimes \mathbf{e}_0' & \tau^{(a)}(\mathbf{u}) \otimes \mathbf{e}_0' \\ -\mathbf{e}_0^* \otimes \sigma^{(a)}(\mathbf{u})\mathbf{u} & \sigma^{(a)}(\mathbf{u}) \end{pmatrix},$$

where $\mathbf{u} = \mathbf{u}_{ab}$. The functions (6.1) must satisfy the properties:

$$\pi_a \circ \sigma^{(a)} = \mathcal{R}_a, \tag{6.2}$$

$$\Delta^{(a)}(\mathbf{0}) = 1 \quad ; \quad \tau^{(a)}(\mathbf{0}) = 0 \quad ; \quad \sigma^{(a)}(\mathbf{0}) = \mathrm{id}_{\mathbf{\Sigma}_a}. \tag{6.3}$$

Property 3 is nothing but a compatibility condition between the mappings of $\mathcal{R}$. It implies that if a rest frame mapping, $\mathcal{R}_a : C_a \to I$, is given, then any other mapping of the family $\mathcal{R}$ can be deduced from $\mathcal{R}_a$ via the cocycle relations (3.4) and (3.5). More precisely, if $b \in I$ is any inertial frame, then the constitutive functions determined by the rest frame mapping $\mathcal{R}_b : C_b \to I$:

$$\Delta^{(b)} : C_b \to \mathrm{I\!R}^+ \quad ; \quad \tau^{(b)} : C_b \to \mathbf{\Sigma}_b^* \quad ; \quad \sigma^{(b)} : C_b \to L_b,$$

must be related to the functions (6.1) by the equations:

$$\begin{aligned}
\Delta^{(b)}(\mathbf{v}') &= (\Delta^{(a)}(\mathbf{u}) + < \tau^{(a)}(\mathbf{u}) \mid \mathbf{u} >)^{-1}(\Delta^{(a)}(\mathbf{v}) + < \tau^{(a)}\mathbf{v}) \mid \mathbf{u} >) \\
\tau^{(b)}(\mathbf{v}') &= (\Delta^{(a)}(\mathbf{u})\tau^{(a)}(\mathbf{v}) - \Delta^{(a)}(\mathbf{v})\tau^{(a)}(\mathbf{u}))\circ \\
&\qquad \circ (\Delta^{(a)}(\mathbf{u})\sigma^{(a)}(\mathbf{u}) + \tau^{(a)}(\mathbf{u}) \otimes \sigma^{(a)}(\mathbf{u})\mathbf{u})^{-1} \\
\sigma^{(b)}(\mathbf{v}') &= \sigma^{(a)}(\mathbf{v}) \circ (\Delta^{(a)}(\mathbf{u})\mathrm{id}_{\mathbf{\Sigma}_a} + \tau^{(a)}(\mathbf{u}) \otimes (\mathbf{v}))\circ \\
&\qquad \circ (\Delta^{(a)}(\mathbf{u})\sigma^{(a)}(\mathbf{u}) + \tau^{(a)}(\mathbf{u}) \otimes \sigma^{(a)}(\mathbf{u})\mathbf{u})^{-1},
\end{aligned} \tag{6.4}$$

where $\mathbf{v}' \in C_b$, $\mathbf{u} = \mathbf{u}_{ab} = \mathcal{R}_a^{-1}(b)$ and $\mathbf{v} = \Phi_{ab}(\mathbf{v}')$.

The assignment of a rest frame map family on the class I is completely equivalent to the assignment of the constitutive functions (6.1) for any $a \in I$. Indeed, it is a simple matter to prove the following theorem:

Theorem 7: *Let I be a class of inertially equivalent reference frames, and let:*

$$\{\Delta^{(a)}, \tau^{(a)}, \sigma^{(a)} / a \in I\}$$

be a family of functions of the type (6.1). If these functions fulfil the properties (6.3) and (6.4) and if, in addition, the mappings:

$$\mathcal{R}_a =: \pi_a \circ \sigma^{(a)} : C_a \to I \quad , \quad a \in I$$

are one-to-one, then $\mathcal{R}$ is a rest frame map family based on I. □

We conclude by defining a *theory of space and time compatible with the inertia principle* as a triplet,

$$\mathcal{T} =: (M; I; \mathcal{R}),$$

consisting of: the universe of events, M; a class of inertially equivalent reference frames of M, I; a rest frame map family based on I, $\mathcal{R}$. All the theories of space and time which

are compatible with the inertia principle can be obtained, as examples of this definition, by specifying the rest frame map family $\mathcal{R}$ - i.e. by specifying the constitutive functions (6.1).

Example 1: Galileian relativity and special relativity. Let's suppose that a theory $\mathcal{T}$ is such that, for a given frame $a \in I$, all the space-time transformations ϕ_{ba} fulfil the reciprocity conditions of section 5. Now, since any ϕ_{ba} is determined by the four parameters in (5.7), then the constitutive functions of the rest frame mapping $\mathcal{R}_a$ are determined by specifying the four real-valued functions:

$$\Delta^{(a)}, \lambda^{(a)}, \mu^{(a)} : C_a \to \mathbb{R}^+ \quad ; \quad \theta^{(a)} : C_a \to \mathbb{R}, \tag{6.5}$$

with the property that:

$$\Delta^{(a)}(\mathbf{0}) = \lambda^{(a)}(\mathbf{0}) = \mu^{(a)}(\mathbf{0}) =: 1 \quad ; \quad \theta^{(a)}(\mathbf{0}) =: 0.$$

Now, Galilean relativity is obtained by assuming:

$$C_a =: \mathbf{\Sigma}_a \tag{6.6}$$

and

$$\begin{aligned} \Delta^{(a)}(\mathbf{u}) = \lambda^{(a)}(\mathbf{u}) = \mu^{(a)}(\mathbf{u}) =: 1 \\ \theta^{(a)}(\mathbf{u}) =: 0. \end{aligned} \tag{6.7}$$

It is a simple matter to show, as a consequence of (6.4), that reciprocity conditions and equations (6.6)-(6.7) hold for any other reference frame of the class I. Similarly, special relativity is obtained by assuming:

$$C_a =: \{\mathbf{u} \in \mathbf{\Sigma}_a / u < c\} \tag{6.8}$$

and:

$$\begin{aligned} \Delta^{(a)}(\mathbf{u}) = \lambda^{(a)}(\mathbf{u}) &=: \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \\ \mu^{(a)}(\mathbf{u}) &=: 1 \\ \theta^{(a)}(\mathbf{u}) &=: -\frac{1}{c^2}\left(1 - \frac{u^2}{c^2}\right)^{-1/2}. \end{aligned} \tag{6.9}$$

Also for these functions it is posible to show, as a consequence of (6.4), that reciprocity conditions and equations (6.6)-(6.7) hold for any other reference frame of the class I.

Example 2: Lorentz relativity and the absolute theory kinematically equivalent to special relativity. Any theory $\mathcal{T}$, based on the existence of the absolute frame, $a \in I$, can be obtained by assuming, as in the previous example, that all the transformations ϕ_{ba} fulfil the reciprocity. Now the Lorentz's relativity [23] is obtained by assuming:

$$C_a =: \{\mathbf{u} \in \mathbf{\Sigma}_a / u < c\} \tag{6.10}$$

and

$$\begin{aligned} \Delta^{(a)}(\mathbf{u}) = \lambda^{(a)}(\mathbf{u}) &=: \gamma(u)\left(1-\frac{u^2}{c^2}\right)^{-1/2} \\ \mu^{(a)}(\mathbf{u}) &=: \gamma(u) \\ \theta^{(a)}(\mathbf{u}) &=: -\frac{1}{c^2}\gamma(u)\left(1-\frac{u^2}{c^2}\right)^{-1/2}, \end{aligned} \tag{6.11}$$

where $\gamma : [0, c) \to \mathbb{R}^+$ is a monotonic function such that $\gamma(0) = 1$; while the absolute theory kinematically equivalent to special relativity [22], [25-27], [29], [30], is obtained by assuming C_a as in (6.10) and:

$$\begin{aligned} \Delta^{(a)}(\mathbf{u}) &=: \left(1-\frac{u^2}{c^2}\right)^{1/2} \\ \lambda^{(a)}(\mathbf{u}) &=: \left(1-\frac{u^2}{c^2}\right)^{-1/2} \\ \mu^{(a)}(\mathbf{u}) &=: 1 \\ \theta^{(a)}(\mathbf{u}) &=: 0. \end{aligned} \tag{6.12}$$

It is a consequence of equations (6.4) that in both these theories the reciprocity conditions for a space-time transformation ϕ_{cb}, with $c \neq a, b \neq a$, and such that $\mathbf{u}_{ac}$ is not proportional to $\mathbf{u}_{ab}$, are not fulfilled.

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References.

[1] J. B. Barbour, ABSOLUTE OR RELATIVE MOTION?, Cambridge Univ. Press, Cambridge, 1989.

[2] P. W. Bridgman, THE LOGIC OF MODERN PHYSICS, The Macmillan Company, New York, 1927.

[3] E. Cassirer, ZUR EINSTEIN'SCHEN RELATIVITATSTHEORIE, Bruno Cassirer Verlag, Berlin, 1920.

[4] I. Ciufolini, J. A. Wheeler, GRAVITATION AND INERTIA, Princeton Univ. Press, Princeton, 1995.

[5] A. Grünbaum, PHILOSOPHICAL PROBLEMS OF SPACE AND TIME, Reidel, Dordrecht, 1973.

[6] M. Jammer, CONCEPTS OF SPACE. THE HISTORY OF THE THEORIES OF SPACE IN PHYSICS, Harvard Univ. Press, Cambridge, Mass., 1954.

[7] A. Koiré, NEWTONIAN STUDIES, Harvard Univ. Press, Cambridge, Mass., 1965.

[8] A. Koiré, ETUDES GALILEENNES, Hermann, Paris, 1966.

[9] P. J. E. Peebles, PRINCIPLES OF PHYSICAL COSMOLOGY, Princeton Univ. Press, Princeton, 1993.

[10] H. Reichenbach, THE PHILOSOPHY OF SPACE AND TIME, Dover, New York, 1958.

[11] M. Schlick, RAUM UND ZEIT IN DER GEGENWARTIGEN PHYSIK, Springer Verlag, Berlino,1922.

[12] R. Torretti, PHILOSOPHY OF GEOMETRY FROM RIEMANN TO POINCARE', Reidel, Dordrecht, 1978.

[13] R. Torretti, RELATIVITY AND GEOMETRY, Pergamon Press, Oxford, 1983.

[14] A. Trautman, in A. Trautman, F. A. E. Pirani, H. Bondi, LECTURES ON GENERAL RELATIVITY, Brandeis 1964 Summer Institute on Theoretical Physics, Prentice-Hall, Englewood Cliff, N. J., 1965.

[15] H. Weyl, SPACE-TIME-MATTER, Dover, New York, 1952.

[16] H. Weyl, PHILOSOPHY OF MATHEMATICS AND NATURAL SCIENCE, Dover, New York, 1963.

[17] E. Whittaker, A HISTORY OF THE THEORIES OF AETHER AND ELECTRICITY, Dover, New York, 1989.

[18] V. Berzi, V. Gorini, *Reciprocity principle and Lorentz transformations*, J. Math. Phys., **10** (1969),1518-1524.

[19] A. Einstein, *Zur Elektrodynamik bewegter Körper*, Ann. Phys., **17** (1905), 891-921.

[20] P. Havas, *Four dimensional formulations of Newtonian mechanics and their relation to the special and the general theory of relativity*, Rev. Mod. Phys., **36** (1964), 938-965.

[21] H. P. Künzle, *Galilei and Lorentz structures in space-time: Comparison of the corresponding geometry and physics*, Ann. Inst. H. Poincaré, **XVII** (1972), 337-362.

[22] R. Mansouri, R. U. Sexl, *A test theory of special relativity*, I,II,III, Gen. Rel. Grav., **8**, (1977), 497-537.

[23] J. H. Poincaré, *Sur la dynamique de l'electron*, Rend. Circ. Mat. Palermo, **21**, (1906), 129-175.

[24] W. A. Rodrigues Jr., Q. A. E. de Souza, Y. Bozhkov, Found. Phys., **25**, (1995), 871-924.

[25] F. Selleri, *Space, time and their transformations*, Chin. J. Eng. Electron., **6**, (1995), 24-44.

[26] F. Selleri, *Noninvariant one-way velocity of light*, Found. Phys., **26**, (1996), 641-664.

[27] F. Selleri, *Noninvariant one-way velocity of light and particle collision*, Found. Phys. Lett., **9**, (1996), 43-60.

[28] G. Spavieri, *Nonequivalence of ether theories and special relativity*, Phys. Rev. A, **34**, (1986), 1708-1713.

[29] A. A. Ungar, *Formalism to deal with Reichenbach's special theory of relativity*, Found. Phys., **21**, (1991), 691-726.

[30] Y. Z. Zhang, *Test theories of special relativity*, Gen. Rel. Grav., **27**, (1995), 475-493.